

PARK–TARTER MATRIX FOR A DYON–DYON SYSTEM

L. G. Mardoyan¹, A. N. Sissakian, V. M. Ter–Antonyan²

Abstract

The problem of separation of variables in a dyon–dyon system is discussed. A linear transformation is obtained between fundamental bases of this system. Comparison of the dyon–dyon system with a 4D isotropic oscillator is carried out.

¹E-mail: mardoyan@thsun1.jinr.dubna.su

²E-mail: terant@thsun1.jinr.dubna.su

1 Introduction

In this paper, we have calculated the matrix between the spherical and parabolic bases of a dyon–dyon system [1] belonging to the same energy level. This matrix is a generalization of the Park–Tarter matrix known from the theory of hydrogen atom [2, 3] to the case when the Coulomb center carries not only the electric but also magnetic charge. Like the Park–Tarter matrix, our matrix is expressed through the Clebsch–Gordan coefficients $C_{a\alpha;b\beta}^{c\gamma}$, however, in our case $a \neq b$, in contrast to the case of a hydrogen atom. We have also traced the connection of the dyon–dyon problem with that of a 4-dimensional isotropic oscillator. As is known [4], these problems are related to each other by the Kustaanheimo–Stiefel transformation [5] supplemented with the 4th (angular) coordinate. We have shown that the coefficients $C_{a\alpha;b\beta}^{c\gamma}$ coincide with the ones [6] of the expansion of the double polar basis over the Euler basis of a 4-dimensional isotropic oscillator.

2 Dyon–Dyon System

A dyon–dyon system in the space \mathbb{R}^3 is described by the equation

$$\left[\left(\frac{\partial}{\partial x_j} - \frac{ie}{\hbar c} A_j \right)^2 - \frac{s^2}{r^2} \right] \psi + \frac{2M_0}{\hbar^2} \left(\epsilon^s + \frac{e^2}{r} \right) \psi = 0 \quad (1)$$

where

$$A_j = \frac{gx_3}{r(r^2 - x_3^2)} (-x_2, x_1, 0)$$

and $s = eg/\hbar c = 0, \pm 1/2, \pm 1, \dots$. Each value of s describes its particular dyon–dyon system. At $s = 0$, eq.(1) is reduced to the Schrödinger equation for a hydrogen atom. When $s \neq 0$, equation (1) preserves O(4)-symmetry and therefore variables in it are separated into spherical, parabolic, and prolate spheroidal coordinates [1].

The system (1) possesses a singularity on the axis x_3 . It is also possible to consider systems with singularities either on the semiaxis $x_3 > 0$ or on $x_3 < 0$, i.e. they are described by the vector potentials

$$A_j^{(\pm)} = \frac{g}{r(r \mp x_3)} (\mp x_2, \pm x_1, 0)$$

and are connected with the system (1) by the gauge transformations

$$A_j^{(\pm)} = A_j + \frac{\partial f^{(\pm)}}{\partial x_j}, \quad \psi^{(\pm)}(\vec{x}) = \psi(\vec{x}) \exp \left(\frac{ie}{\hbar c} f^{(\pm)} \right)$$

with the gauge function $f^{(\pm)} = \pm 2g \arctan x_2/x_1$.

The variables in eq. (1) are separated in spherical and parabolic coordinates.

In the spherical coordinates

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta \quad (2)$$

the wave function of the dyon–dyon system is of the form [7]

$$\psi_{nkm}^{(s)}(r, \theta, \varphi) = R_{nkm}^{(s)}(r) Z_{km}^{(s)}(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

where the functions $Z_{km}^{(s)}(\theta)$ and $R_{nkm}^{(s)}(r)$ normalized by the condition

$$\int_0^\pi \sin \theta Z_{k'm}^{(s)}(\theta) Z_{km}^{(s)}(\theta) d\theta = \delta_{k'k}, \quad \int_0^\infty r^2 [R_{nkm}^{(s)}(r)]^2 dr = 1$$

are given by the formulae

$$Z_{km}^{(s)}(\theta) = N_{km}^{(s)} (1 - \cos \theta)^{\frac{|m-s|}{2}} (1 + \cos \theta)^{\frac{|m+s|}{2}} P_k^{(|m-s|, |m+s|)}(\cos \theta)$$

$$R_{nkm}^{(s)}(r) = C_{nkm}^{(s)} \exp\left(-\frac{r}{r_0 n}\right) \left(\frac{2r}{r_0 n}\right)^{k + \frac{|m-s| + |m+s|}{2}}$$

$$F\left(-n + k + \frac{|m-s| + |m+s|}{2} + 1; 2k + |m-s| + |m+s| + 2; \frac{2r}{r_0 n}\right)$$

Here $P_n^{(\alpha, \beta)}(x)$ are Jacobi polynomials; $r_0 = \hbar^2/M_0 e^2$ is the Bohr radius. The normalization constants $N_{km}^{(s)}$ and $C_{nkm}^{(s)}$ equal

$$N_{km}^{(s)} = \left[\frac{(2k + |m-s| + |m+s| + 1)k!(k + |m-s| + |m+s|)!}{2^{|m-s| + |m+s| + 1} \Gamma(k + |m-s| + 1) \Gamma(k + |m+s| + 1)} \right]^{1/2}$$

$$C_{nkm}^{(s)} = \frac{2}{n^2 r_0^{3/2}} \frac{1}{(2k + |m-s| + |m+s| + 1)!} \sqrt{\frac{(n + k + \frac{|m-s| + |m+s|}{2})!}{(n - k - \frac{|m-s| + |m+s|}{2} - 1)!}}$$

Quantum numbers run over the values $n = 1, 3/2, 2, \dots, k = 0, 1, \dots, k_{max}$, where

$$k_{max} = n - \frac{|m-s| + |m+s|}{2} - 1$$

The energy spectrum of the system is of the form

$$\epsilon_n^s = -\frac{M_0 e^4}{2\hbar^2 n^2} \quad (3)$$

In the parabolic coordinates

$$x_1 = \sqrt{\xi\eta} \cos \varphi, \quad x_2 = \sqrt{\xi\eta} \sin \varphi, \quad x_3 = \frac{1}{2}(\xi - \eta) \quad (4)$$

upon the substitution

$$\psi(\xi, \eta, \varphi) = f_1(\xi) f_2(\eta) \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

the variables in (1) are separated, which results in the system of equations

$$\begin{aligned}\frac{d}{d\xi} \left(\xi \frac{df_1}{d\xi} \right) + \left[\frac{M_0 \epsilon^s}{2\hbar^2} \xi - \frac{(m+s)^2}{4\xi} + \beta_1 \right] f_1 &= 0 \\ \frac{d}{d\eta} \left(\eta \frac{df_2}{d\eta} \right) + \left[\frac{M_0 \epsilon^s}{2\hbar^2} \eta - \frac{(m-s)^2}{4\eta} + \beta_2 \right] f_2 &= 0\end{aligned}$$

where

$$\beta_1 + \beta_2 = \frac{M_0 e^2}{\hbar^2} \quad (5)$$

At $s = 0$, these equations coincide with the equations for a hydrogen atom in the parabolic coordinates [8], and consequently,

$$\psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi) = \frac{\sqrt{2}}{n^2 r_0^{3/2}} f_{n_1, m+s}(\xi) f_{n_2, m-s}(\eta) \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

where

$$f_{pq}(x) = \frac{1}{\Gamma(|q|+1)} \sqrt{\frac{\Gamma(p+|q|+1)}{p!}} \exp\left(-\frac{x}{2r_0 n}\right) \left(\frac{x}{r_0 n}\right)^{\frac{|q|}{2}} F\left(-p; |q|+1; \frac{x}{r_0 n}\right)$$

Here n_1 and n_2 are non-negative integers

$$n_1 = -\frac{|m+s|+1}{2} + \frac{\hbar}{\sqrt{-2M_0\epsilon^s}}\beta_1, \quad n_2 = -\frac{|m-s|+1}{2} + \frac{\hbar}{\sqrt{-2M_0\epsilon^s}}\beta_2$$

from which and (3), (5) it follows that the parabolic quantum numbers n_1, n_2, m and s are connected with the principal quantum number n as follows:

$$n = n_1 + n_2 + \frac{|m-s|+|m+s|}{2} + 1 \quad (6)$$

3 Park–Tarter Generalized Matrix

We write the searched expansion in the form

$$\psi_{n_1 n_2 m}^{(s)}(\xi, \eta, \varphi) = \sum_{k=0}^{k_{max}} T_{n_1 n_2 k m}^{(s)} \psi_{n k m}^{(s)}(r, \theta, \varphi) \quad (7)$$

Our purpose is to calculate the coefficients $T_{n_1 n_2 k m}^{(s)}$, i.e. the Park–Tarter generalized matrix. The usual Park–Tarter matrix is the matrix $T_{n_1 n_2 k m}^{(s)}$ at $s = 0$.

We substitute

$$\xi = r(1 + \cos \theta), \quad \eta = r(1 - \cos \theta),$$

into the left-hand side of expansion (7), let r tend to infinity, take the formula

$$F(-n; c; x) \sim (-1)^n \frac{\Gamma(c)}{\Gamma(c+n)} x^n, \quad (x \rightarrow \infty)$$

and the orthogonality condition for the function $Z_{km}^{(s)}$ into account. All this leads to the formula

$$T_{n_1 n_2 km}^{(s)} = (-1)^k B_{n_1 n_2 km}^{(s)} I_{n_1 n_2 km}^{(s)}$$

where

$$B_{n_1 n_2 km}^{(s)} = \sqrt{\frac{(2k + |m - s| + |m + s| + 1)k!(k + |m - s| + |m + s|)!}{2^{2n + |m - s| + |m + s|} \Gamma(k + |m - s| + 1) \Gamma(k + |m + s| + 1)}} \left[\frac{\left(n - k - \frac{|m - s| + |m + s|}{2} - 1\right)! \left(n + k + \frac{|m - s| + |m + s|}{2}\right)!}{(n_1)!(n_2)! \Gamma(n_1 + |m + s| + 1) \Gamma(n_2 + |m - s| + 1)} \right]^{1/2}$$

and the second factor is equal to the integral

$$I_{n_1 n_2 km}^{(s)} = \int_{-1}^1 (1 - x)^{n_2 + |m - s|} (1 + x)^{n_1 + |m + s|} P_k^{(|m - s|, |m + s|)}(x) dx$$

Then taking advantage of the Rodrigues formula [9]

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d^n}{dx^n} \left[(1 - x)^{\alpha+n} (1 + x)^{\beta+n} \right]$$

and the integral representation for the Clebsch–Gordan coefficients [10]

$$C_{a\alpha; b\beta}^{c\gamma} = \delta_{\alpha+\beta=\gamma} \left[\frac{(2c + 1)(J + 1)!(J - 2c)!(c + \gamma)!}{(J - 2a)!(J - 2b)!(a - \alpha)!(a + \alpha)!(b - \beta)!(b + \beta)!(c - \gamma)!} \right]^{1/2} \frac{(-1)^{a-c+\beta}}{2^{J+1}} \int_{-1}^1 (1 - x)^{a-\alpha} (1 + x)^{b-\beta} \frac{d^{c-\gamma}}{dx^{c-\gamma}} \left[(1 - x)^{J-2a} (1 + x)^{J-2b} \right] dx$$

($J = a + b + c$), we obtain

$$T_{n_1 n_2 ms}^{nk} = (-1)^{n_2+k} C_{a\alpha; b\beta}^{c\gamma} \quad (8)$$

where

$$a = \frac{n_1 + n_2 + |m + s|}{2}, \quad b = \frac{n_1 + n_2 + |m - s|}{2}, \quad c = k + \frac{|m - s| + |m + s|}{2}$$

$$\alpha = \frac{n_1 - n_2 + |m + s|}{2}, \quad \beta = \frac{n_2 - n_1 + |m - s|}{2}, \quad \gamma = \frac{|m - s| + |m + s|}{2}$$

At $s = 0$ formula (8) turns into the Park–Tarter formula, as would be expected.

4 Dyon–Dyon System and 4D Oscillator

Let us demonstrate that if in eq. (1) we make the changes

$$s \rightarrow -i \frac{\partial}{\partial \gamma}, \quad \psi(\vec{x}) \rightarrow \psi(\vec{x}, \gamma) = \psi(\vec{x}) \frac{e^{is\gamma}}{\sqrt{4\pi}} \quad (9)$$

($\gamma \in [0, 4\pi)$), it will transform into the Schroedinger equation for a 4D isotropic oscillator.

Equation (1) in the spherical coordinates is of the form

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] - \\ \frac{2is \cos \theta}{r^2 \sin^2 \theta} \frac{\partial \psi}{\partial \varphi} - \frac{s^2}{r^2 \sin^2 \theta} \psi + \frac{2M_0}{\hbar^2} \left(\epsilon^s + \frac{e^2}{r} \right) \psi = 0 \end{aligned} \quad (10)$$

From (9) and (10) we have

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\hat{J}^2}{r^2} \right] \psi + \frac{2M_0}{\hbar^2} \left(\epsilon^s + \frac{e^2}{r} \right) \psi = 0 \quad (11)$$

where

$$\hat{J}^2 = - \left[\frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \left(\sin \beta \frac{\partial}{\partial \beta} \right) + \frac{1}{\sin^2 \beta} \left(\frac{\partial^2}{\partial \alpha^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right) \right]$$

Here we change the notation: $\beta = \theta$ and $\alpha = \varphi$. If we now pass from the coordinates r, α, β, γ to the coordinates

$$u_0 + iu_1 = u \cos \frac{\beta}{2} e^{-i \frac{\alpha + \gamma}{2}}, \quad u_2 + iu_3 = u \sin \frac{\beta}{2} e^{i \frac{\alpha - \gamma}{2}} \quad (12)$$

with $u^2 = r$, take into account that

$$\frac{\partial^2}{\partial u_\mu^2} = \frac{1}{u^3} \frac{\partial}{\partial u} \left(u^3 \frac{\partial}{\partial u} \right) - \frac{4}{u^2} \hat{J}^2$$

and introduce the notation

$$E = 4e^2, \quad \epsilon^s = -\frac{M_0 \omega^2}{8}$$

then equation (11) will turn into the Schrödinger equation for a 4D isotropic oscillator

$$\left[\frac{\partial^2}{\partial u_\mu^2} + \frac{2M_0}{\hbar} \left(E - \frac{M_0 \omega^2 u^2}{2} \right) \right] \psi(\vec{u}) = 0$$

whose energy spectrum is given by the formula

$$E_N = \hbar \omega (N + 2) \quad (13)$$

Introducing the double polar coordinates

$$u_0 + iu_1 = \rho_1 e^{-i\varphi_1}, \quad u_2 + iu_3 = \rho_2 e^{i\varphi_2} \quad (14)$$

from formulae (2), (4), (12), and (14) we get the relations

$$\xi = 2\rho_1^2, \quad \eta = 2\rho_2^2, \quad \varphi = \varphi_1 + \varphi_2, \quad \gamma = \varphi_1 - \varphi_2$$

which lead to the formulae

$$\psi_{NJM_1M_2}(u, \alpha, \beta, \gamma) = 4n\sqrt{\frac{2}{\lambda}} \delta_{n, \frac{N}{2}+1} \delta_{k, J - \frac{|M_1 - M_2| + |M_1 + M_2|}{2}} \delta_{m, M_1} \delta_{s, M_2} \psi_{nkms}(r, \theta, \varphi, \gamma)$$

$$\psi_{N_1N_2m_1m_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) = 4n\sqrt{\frac{2}{\lambda}} \delta_{n_1, N_1} \delta_{n_2, N_2} \delta_{m, \frac{m_1 + m_2}{2}} \delta_{s, \frac{m_1 - m_2}{2}} \psi_{n_1n_2ms}(\xi, \eta, \varphi, \gamma)$$

generalizing the earlier results [6, 11].

Now we are able to write the expansion [6]

$$\psi_{N_1N_2m_1m_2}(\rho_1, \rho_2, \varphi_1, \varphi_2) = \sum_{J=J_{min}}^{N/2} W_{N_1N_2m_1m_2}^{NJM_1M_2} \psi_{NJM_1M_2}(u, \alpha, \beta, \gamma) \quad (15)$$

where

$$W_{N_1N_2m_1m_2}^{NJM_1M_2} = e^{i\pi\Phi} C_{a_0, \alpha_0; b_0, \beta_0}^{c_0, \gamma_0} \quad (16)$$

$$a_0 = \frac{N + |m_1| - |m_2|}{4}, \quad b_0 = \frac{N - |m_1| + |m_2|}{4}, \quad c_0 = J$$

$$\alpha_0 = \frac{N + |m_1| - |m_2|}{4} - N_2, \quad \beta_0 = \frac{N - |m_1| + |m_2|}{4} - N_1, \quad \gamma_0 = \frac{|m_1| + |m_2|}{2}$$

The lower limit of summation in (15) and quantity Φ are given by the expressions

$$J_{min} = \frac{1}{2} (|M_1 - M_2| + |M_1 + M_2|)$$

$$\Phi = N_2 + J - \frac{|m_1| + |m_2|}{2} - \frac{m_2 + |m_2|}{2}$$

We conclude with the following two comments:

(a) Using formulae (2) and (12) and considering that $r = u^2, \theta = \beta, \varphi = \alpha$, one can easily show that

$$x_1 = 2(u_0u_2 + u_1u_3)$$

$$x_2 = 2(u_0u_3 - u_1u_2)$$

$$x_3 = u_0^2 + u_1^2 - u_2^2 - u_3^2$$

$$\gamma = \frac{i}{2} \ln \frac{(u_0 + iu_1)(u_2 + iu_3)}{(u_0 - iu_1)(u_2 - iu_3)}$$

The first three lines are the transformation $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ suggested by Kustaanheimo and Stiefel for the regularization of equations of celestial mechanics [5]. Later, this transformation found other applications, as well [12, 13]. This transformation supplemented with the coordinate γ was used for the "synthesis" of the dyon–dyon system from the 4D isotropic oscillator [4].

(b) It is known [6] that diagonal ($m_1 = m_2$) elements of the matrix $W_{N_1 N_2 m_1 m_2}^{N J M_1 M_2}$ with N even coincide with the Park–Tarter matrix. From formula (16) it follows that the remaining elements of the matrix $W_{N_1 N_2 m_1 m_2}^{N J M_1 M_2}$ have also a physical meaning: these are elements of the generalized Park–Tarter matrix for the dyon–dyon system.

5 Degeneracy of the Energy Levels

Let us discuss the problem of multiplicity of degeneration of the energy levels (3) and (13). From formula (6) it follows that at fixed n, m and s the energy levels are degenerate with the multiplicity

$$g_{nm}^s = n - \frac{|m - s| + |m + s|}{2}$$

For $s \geq 0$ the multiplicity of degeneration of levels (3) at fixed s and n is

$$g_n^s = \sum_{|m| \geq [s]} g_{nm}^s + \sum_{|m| \leq [s]-1} g_{nm}^s$$

where the upper limit of summation is determined from the condition $g_{nm}^s \geq 0$,

$$|m - s| + |m + s| \leq 2n - 2$$

Therefore,

$$g_n^s = \sum_{m=-[s]+1}^{[s]-1} (n - s) + 2 \sum_{m=[s]}^{[n]-1} (n - m) = (n - s)(n + s) \quad (17)$$

The same result follows from analogous computations also when $s < 0$.

The quantum numbers s and n in formula (17) assume simultaneously either integer or half-integer values, and thus, we have

$$g_n = \sum_{s=-n+1}^{n-1} g_n^s = \frac{1}{3}n(2n - 1)(2n + 1)$$

where g_n stands for the multiplicity of degeneration of the energy levels (13) of the 4d oscillator. Since $N = 2n - 2$, we arrive at the known result

$$g_N = \frac{1}{6}(N + 1)(N + 2)(N + 3)$$

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